Summer research report

Yixuan Shi

November 14, 2011

In the last summer, I did some research reading work under the supervision of Professor Jenny Li. I got really great support from Professor Li and Women In Mathematics (WIM).Thanks to the financial support I recieved from WIM that I can fully focused on my acedemic reseach and benefited a great deal from this amazing opportunity.
The following is a general report of my summer research:
In the beginning of summer, I studied the paper 'A monetary Equilibrium Model with Transactions Costs' by Julio J. Rotemberg.
Professor Rotemberg presented the competitive equilibrium of an economy in which people hold money for transactions purposes with transaction cost. In the model, we split the lifetime budget constraint, utility function and money balance into two parts, day-1 which includes households exchange capital for money and day-2 for households to consume purely without any transaction.
There are three parties involved in our model: households, financial intermediaries and Government.

For 2n households,at time t, the households are assumed to maximize the utlity function given by:

$$V_t = \sum_{\tau=t}^{\infty} \rho^{\tau-t} lnC_{\tau}^{i}$$

Suppose that household i engages in financial transactions in the "even" periods, t, t+2,t+4,...At these dates it withdraws an amount M_{τ}^{i} of money balances that must be sufficient to pay for its consumption at τ and $\tau+1$:

$$M_{\tau}^{i} = P_{\tau}C_{\tau}^{i} + P_{\tau+1}C_{\tau+1}^{i}$$

With transformations and mathematical computation, we may reach the lifetime utlity function as following:

$$V_t = \sum_{k=0}^{\infty} \rho^{2k}[(1+\rho)\ln\left(\frac{M_{\tau+k}^{i}}{P_{\tau+k}}\right) + \rho\ln\rho - (1+\rho)\ln(1+\rho) - \rho\ln\left(\frac{P_{t+1+k}}{P_{t+k}}\right)]$$

This expression must be maximized with respect to the sequence of monetary withdrawals subject to the lifetime budget constraint of the household at t:

$$\sum_{\tau=0}^{\infty} \frac{M^i_{t+2\tau}}{P_{t+2\tau}} / \prod_{i=0}^{2\tau-1} (1 + r_{t+i})$$

$$= \left[\sum_{\tau=0}^{\infty} Y^i_{t+\tau} / \prod_{i=0}^{\infty} (1 + r_{t+i}) \right] + K^i_{t-1}(1 + r_{t-1}) - \left[\sum_{\tau=0}^{\infty} B / \prod_{i=0}^{\infty} (1 + r_{t+i}) \right]$$

For government meanwhile, in this model, has no expenditures. Instead of that, it levies taxes, issues money, and holds capital. We may find the increase of government capital is given by:

$$K^G_{\tau+1} = f'(K_\tau) K^G_\tau + \frac{M_{\tau+1} - M_\tau}{P_{\tau+1}} + T_{\tau+1} \tag{0.1}$$

where $T_{\tau+1}$ is the real taxes levied at $\tau + 1$, K^G_τ is the government's real holdings of capital at τ and M_τ is high-powered money at τ.

As for financial intermediaries, they receive the household's income and invest it in claims on capital. Besides, they issue certain quantity of money and been compensated for their services with the brokerage fees, B, shown in households budget constraint

The equilibrium we are seeking for is a path for the price level and for the real rate of interest such that households maximize utility and production firms maximize profits under these prices and following conditions hold:
1. Affodability: the sum of consumption and capital demanded at τ by the households and capital demanded by the government at $\tau$is equal to output at τ:

$$C_\tau + K^P_\tau + K^G_\tau = \overline{L} f(\frac{K_{t-1}}{\overline{L}}).$$

2. The amount of money that households that visit the intermediaries at τ want to hold between τ and $\tau + 1$ must be equal to M_τ

With aggregate consumption condition and transversality condition, we obtains the difference equation that governs the evolution of aggregate capital:

$$\overline{L} f\Big(\frac{K_{\tau+2}}{\overline{L}}\Big) - K_{\tau+3} =$$

$$\rho^2 \frac{1 + \rho(M_{\tau+2}/M_{\tau+3})}{1 + \rho(M_\tau/M_{\tau+1})} f'\Big(\frac{K_{\tau+1}}{\overline{L}}\Big) f'\Big(\frac{K_{\tau+2}}{\overline{L}}\Big) \times \left[\overline{L} f\Big(\frac{K_\tau}{\overline{L}}\Big) - K_{\tau+1} \right],$$

$$\tau = t - 1,\ t,\ t + 1, \ldots$$

With knowledge of the sequence of capitals provides the seuqence of rate of return, the aggregate consumption and sequence of individual consumption, and the sequence of prices, the equilibrium is thus a third-order nonlinear differential eqution with one initial condition K_{t-1}. And as long as $\frac{M_\tau}{M_{\tau+1}}$ converges to a constant, guarded by transversality condition in Arrow and

Kurz (1970), the only stead state with positive consumption has the property that $\rho f'(\overline{K}) = 1$ However Professor Rotemberg only presented a linearized version around $\overline{K}$ which is:

$$(K_{\tau+3} - \overline{K}) - \Big\{ f'(\overline{K}) - \rho\Big[f\Big(\frac{\overline{K}}{\overline{L}}\Big) - \frac{\overline{K}}{\overline{L}}\Big] f''\Big(\frac{\overline{K}}{\overline{L}}\Big)\Big\}(K_{\tau+2} - \overline{K})$$

$$-\Big\{ 1 - \rho\Big[f\Big(\frac{\overline{K}}{\overline{L}}\Big) - \frac{\overline{K}}{\overline{L}}\Big] f''\Big(\frac{\overline{K}}{\overline{L}}\Big)\Big\}(K_{\tau+1} - \overline{K}) + f'\Big(\frac{\overline{K}}{\overline{L}}\Big)(K_{\tau} - \overline{K}) = 0.$$

or

$$(1 - \lambda_1 L)(1 - \lambda_2 L)(1 - \lambda_3 L)(K_\tau - \overline{K}) = 0$$

where L is the lag operator

$$\begin{aligned}
\lambda_1\lambda_2\lambda_3 &= -f'\Big(\frac{\overline{K}}{\overline{L}}\Big), \\
\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 &= -1 + \rho\Big[f\Big(\frac{\overline{K}}{\overline{L}}\Big) - \frac{\overline{K}}{\overline{L}}\Big] f''\Big(\frac{\overline{K}}{\overline{L}}\Big), \\
\lambda_1 + \lambda_2 + \lambda_3 &= f'\Big(\frac{\overline{K}}{\overline{L}}\Big) - \rho\Big[f\Big(\frac{\overline{K}}{\overline{L}}\Big) - \frac{\overline{K}}{\overline{L}}\Big] f''\Big(\frac{\overline{K}}{\overline{L}}\Big).
\end{aligned}$$

Professor Rotemberg presented the proof of existence of equilibrium but failed to solve the math part which was accomplished by later research 'Non-steady-state equilibrium solution of a class of dynamic models' by Jenny X. Li (2000). Professor Li argued that by using nonlinear differential equations of second or higher order, we can firstly manage to describe and solve the problem numerically and then closed form solution.
We may take Cobb-Douglas production fuction $f(K_t) = K_t^\alpha$ where $0 < \alpha < 1$. Without loss of generality, we assume that the labor supply constant $\overline{L} = 1$. For this type of production function the equilibrium level of capital sequence satisfies:

$$K_{t+2}^\alpha - K_{t+3} = \alpha^2\beta^2 \frac{1 + \beta(M_{t+2})/(M_{t+3})}{1 + \beta(M_t)/(M_{t+1})} K_{t+1}^{\alpha-1} K_{t+2}^{\alpha-1}(K_t^\alpha - K_{t+1})$$

Steady-state equilibrium is:

$$\overline{K} = (\alpha\beta)^{1/(1-\alpha)}$$

The main idea is that we shall look for equilibrium path in a special form, namely

$$K_{t+1} = g(K_t), \quad t \geq t_0$$

for some function g. After re-writing, we may reach the following equivalent form:

$$F(g(x)) = g(x)$$

where

$$F(g(x)) = x^\alpha - (\alpha\beta)^{-2}(g(g(x))^\alpha - g(g(g(x))))g(g(x))^{1-\alpha}g(x)^{1-\alpha}$$

There exists a fixed point of F which equivalently saying is our equilibrium by using contraction mapping theorem.

Furthermore, Professor Li evaluated the change of the equilibrium capital sequence in 2 different cases of non-steady-state equilibrium path, one with money supply increased by 2 percent at first period and the other with money supply increases 1 percent each at both 2 periods.

In this model, Professor Li inspected how market been affected when open-market monetary injection involved. If we want to get one step further, we may asked how market been affected when government levies taxes in certain ways.