Front propagation in reaction- diffusion equations

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Abstract

We investigate the behavior of solutions of reaction-diffusion equations in a heterogeneous medium.

Contents

1 Introduction

In this project, I am studying the logistics equation and the heat equation and also the combination of the two, the reaction-diffusion equation. The logistics equation is a first-order differential equation that is widely used to model the growth of a population. At first, the function begins to grow exponentially. After some time, the curve begins to level off and become asymptotic to a certain number, namely the carrying capacity, k. In our case, the carrying capacity is one. One application of the logistics equation is the study of biological invasion.

Biological invasion can be described as the study of how certain animals and plant species spread across the globe. Relating the logistics equation to biological invasion, we can see that with this equation, a population of a certain species will grow and expand, leveling off at the carrying capacity. Without any diffusion or dispersion, the population of this certain species is confined to one specific living area. For example, let's say there exists a population of mice living on an island in the Pacific Ocean. In this instance, there is no possibility for diffusion since mice cannot cross the ocean without human assistance. Thus, the logistics equation is

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the only means of describing the population growth. The heat equation, however, describes a situation in which there is diffusion

The heat equation is also a differential equation. However, it is a second order differential equation. The heat equation is used to describe the diffusion of heat over time. Generally, given a certain area in space, heat will move from warmer areas to colder ones. Therefore, the warm spots will cool down and the colder spots will begin to warm up. In terms of biological invasion, this equation describes diffusion of animal species without the logistics equation. Sticking with our previous example, let's say a number of mice are living in a forest in Pennsylvania. For some unknown reason, they are unable to effectively reproduce. Let's say that after a certain amount of time, t, their food source begins to grow thin. Due to this, the mice now have to relocate themselves, thus diffusion occurs. The heat, or diffusion, equation describes their movement, as it does not involve the growth of the population. The combination of these two equations, logistics and diffusion, describes a situation in which both dispersion and population growth occur.

The reaction-diffusion equation is a differential equation which accounts for diffusion and growth. It is from this equation that the travelling wave phenomenon is produced. With respect to our mice example, this equation would be used if the mice could both move from the current place they live and also reproduce. If the mice lived in a forest somewhere, with the biological ability to mate, and they ran out of food, they would need to migrate somewhere. This migration is the diffusion part of the equation. Then, as they migrate, they can also reproduce, thus the reaction portion of the equation. In conclusion, biological invasion is one application of the reaction-diffusion equation that I am studying.

Generally, when we use reaction-diffusion equations to model biological invasions, there are many different forms of the equation that we can use. Two that I have studied, the Fisher equation and Skellams equation, are both widely used in the study of biological invasion. Skellams equation,

$$\frac{\partial n}{\partial t} = D(\frac{\partial^2 n}{\partial x^2} + \frac{\partial^2 n}{\partial y^2}) + \epsilon n$$

is a form of the reaction-diffusion equation with some modifications to the logistics part of the equation. Instead of the usual equation,

$$u_x = u(1 - u)$$

there is no carrying capacity in this form. This is sometimes referred to as Malthusian because it will grow without bound. As it applies to biological invasion, this means that in our group of mice, there is virtually no competition. This Skellams equation models a species with no means of death. This is why this part of the equation grows without bound. The Fisher equation, however, does account for competition.

The Fisher equation is of the form

$$\frac{\partial n}{\partial t} = D(\frac{\partial^2 n}{\partial x^2} + \frac{\partial^2 n}{\partial y^2}) + (\epsilon - \mu n)n$$

This equation is very similar to the Skellams equation except for the logistics part of the equation. This part is exactly like the regular logistics equation. We use this Fisher equation when we study biological invasion because it is more realistic. It is possible for some species to have very little to no competition, but it is very unlikely. That is why we rarely use the Skellams equation, and mainly stick to Fisher's equation.

Despite their long term differences, the two solutions are very similar for very small time, t. This happens because in the Skellams equation, there is never competition, and while t is very small, the size of a species is low, so there is minimal competition. However, for large t, the equations are very different and we mainly use the Fisher equation as we study biological invasion.

It is exciting to study heterogeneous media modeled by reaction-diffusion equations of the Fisher-Kolmogorov-Petrovsky-Piskunov (F-KPP) type [3, 2]

$$\partial_t u = (a(x)u_x)_x + f(x,u), \qquad x \in [-10, 10],\ t > 0, \tag{1.1}$$

$$u = g(x),\ \text{at } t = 0, \tag{1.2}$$

$$u = 0,\ \text{when } x = \pm 10. \tag{1.3}$$

$$\tag{1.4}$$

with a and f discontinuous in x. We state our precise assumptions later in Section 1.2. The function $g(x)$ is called the initial condition, and the assumptions $u = 0$ at $x = \pm 10$ are called the (homogeneous Dirichlet) boundary conditions.

1.1 Why do we want to study reaction-diffusion equations

Fisher-KPP type equations arise in many contexts where there is a spatial advancement of one state into propagating into a homogeneous state, such as simple combustion models for flame propagation [5, 8, 9], as well as in models in biology and population dynamics [1, 4, 6, 7].

In these models, the underlying media is heterogeneous at the scale under consideration. In population dynamics for instance, heterogeneities arise since roads, forests, lakes, have ecological different properties and thus must taken account in the model. In the case of combustion, one heterogeneities may arise, for example, if a heterogeneous material is made by adding small inclusions to a bulk material which have different properties. At the scale in which these models are valid, the interface between desert and road, say, is continuous, whereas their properties need not vary continuously across the interface. This leads to a model for which the coefficients modeling the material properties depend discontinuously in the space variable x, but the discontinuities form surfaces with some regularity.

There are many reasons why we would want to study these equations in terms of biological invasion. One reason is so that we know what animals have the ability to travel and migrate to certain places. If we know already that these mice do not have the ability to reproduce, then we know to mainly study the diffusion equation. Similarly, if we know that the mice are on an island miles away from another body of land, then we would know to only study the logistics equation when we study the population growth. Once this equation is known, there are many situations in which this information can be very useful.

Biological invasion can be a good thing, but it can also have very negative effects on both our environment and even our economy. For example, the Brown tree snake, orginially an inhabitant of the South Pacific, was somehow mistakenly transported to the island of Guam. This tree snake is enormous, with the largest recorded length at three meters. Essentially, when this animal was accidentally transported to Guam, it reproduced and the snakes basically destroyed the island. Because the snakes were not native to the island, their place in the food chain was unclear. Many birds that lived in the trees became extinct because the snakes were forced to prey on them. There were many power outages on the island because the snakes would climb the poles. Since this island was not equipt for the arrival of these snakes, there were very few predators to control the population. Essentially the population grew without bound. In this case the equation called Skellam's model is very applicable. This equation nearly identical to the reaction diffusion equation except there is no competition. With this equation, the carrying capacity is virtually zero. This example is a very good reasn why it is important to study these equations. After taking all of the factors into account, we would know that these snakes are going to continue to devestate this island if nothing stops them.

Another example is the current FDA warning against certain types of raw red tomatoes. There are many tomatoes which are infected with salmonella, a very serious and potentially fatal bacteria. Every so often this seems to happen with different vegetables, a year ago the same situation happened with spinach. The reaction-diffusion equation would be very helpful in this situation because it would be able to tell us the range in which this bacteria could travel. If this information had been known prior to the problem, maybe some precautionary messures could have been taken. Unfortunately, these biological invasions can also have an effect on our economy.

This summer, the Olympmic games will be held in Beijing, China. Due to this, many non-native grass seeds and plant seeds are being imported into the country in an effort to create a visually appealing atmosphere for the games. Unfortunately, as these non-native seeds are being imported, these seeds are carrying insects and pests that are also not native to China, and therefore could have very little predators, as in the case of the Brown tree snake. Knowing this, scientists are able to use the reaction-diffusion equations to estimate the potential damage this is going to cause. As of right now, the economic loss is estimated at \$14.5 billion.

In conclusion, there are many reasons why it is important to study these equations. From these equations we can study many migration and population habits of a variety of different species. In this report, I will describe many of the possible solutions to these equations and also how to arrive at such solutions.

1.2 Structure of the Heterogeneous Medium

We assume

$$\Omega_{\text{fluid}} = [-10,-8] \cup [-4,0] \cup [4,8], \;\; \Omega_{\text{solid}} = [-8,-4] \cup [0,4] \cup [8,10] \tag{1.5}$$

and consider solutions of

$$\frac{\partial u}{\partial t} = (a(x)u_x)_x + f(x,u) \text{ in } [-10,10], u = g(x), \text{ at } t=0, u=0, \text{ when } x = \pm 10. \tag{1.6}$$

where

$$f(x,u) = \begin{cases} \beta\ u(1-u),\ x \in \Omega_{fluid}, \\ 0,\ x \in \Omega_{\text{solid}}, \end{cases}$$

$$a(x) = \begin{cases} a_f,\ x \in \Omega_{fluid}, \\ a_s,\ x \in \Omega_{\text{solid}}, \end{cases}$$

and the constants a_f, a_s and β are positive.

2 Effect of different terms in the classical reaction-diffusion equations

Consider the classical reaction diffusion equation:

$$\partial_t u = u_{xx} + u(1-u), \qquad x \in [-10,10]\ t > 0 \tag{2.1}$$

$$u = \begin{cases} 0, x < -1 \\ 1+x, \leq x < 0 \\ 0 \leq x < 1-x \\ x \geq 1. \end{cases} \tag{2.2}$$

$$u = 0,\ \text{when } x = \pm 10. \tag{2.3}$$

Equations (2.1) and (1.1) are the same if $a(x) = 1$ and $f(x,u) = u(1-u)$ and the initial condition

$$g(x) = \begin{cases} 0, x < -1 \\ 1+x, \leq x < 0 \\ 0 \leq x < 1-x \\ x \geq 1. \end{cases} \tag{2.4}$$

$$u = 0,\ \text{when } x = \pm 10. \tag{2.5}$$

2.1 Logisitcs equation

If we ignore the term $(a(x)u_x)_x = u_{xx}$ then we have an ODE:

$$\partial_t u = u(1-u),\ x \in [-10,10],\ t > 0 \tag{2.6}$$

$$u = \begin{cases} 0, x < -1 \\ 1+x, \leq x < 0 \\ 0 \leq x < 1-x \\ x \geq 1. \end{cases} \tag{2.7}$$

$$u = 0,\ \text{when } x = \pm 10. \tag{2.8}$$

Writting this equation into Matlab we obtain the following lines of code:

– beta=1; this assigns the value of 1 to the variable beta; $\beta = 1$

–dh=0.1; this is the spatial distance

–$size_x = 10$; this tells us that $x \in [-10, 10]$

–$nx = round(2 * size_x/dh) + 1$; this is the number of points, we multiply the size of x by two and divide by the number of spaces

–$x = linspace(-size_x, size_x, nx)$; this is the location of the points. The points range from x=-10 to x=10 and nx represents the increment. In this case, nx=201

–$\chi_{solid} = 0 * x'$; this means that for any x, χ_{solid} will be zero. x' means it is the transpose of the x vector

–$dt = .01$; this is the time step

–$T = 10$; this is the terminal time

–$N = round(T/dt)$; This is the number of steps. It is the difference between the terminal time and the time step, rounded to the nearest whole number.

–$u_{out} = 2sinx$; this is the initial condition

–$u = []$; this line makes the matrix that will eventually have all of our values of u in it

–for i=1:1:N; i is the counter, the proccess runs until it reaches N. The 1 stands for the increment. It starts at the value of 1, and increases by one each time until it reaches N.

–$u_{out} = u_{out} + dt * u_{out} + (1 - u_{out}) * (1 - \chi_{solid}) * \beta$

This line is the Logistics Equation. It takes the previous value for u_{out} and adds to it the new value. In the previous lines of code, the assigned the value of 1 to β and 0 to χ_{solid}. Thus this equation is simply $u_{out} = u_{out} + dt * u_{out} + u_{out}(1 - u_{out})$. We obtain this from the following computation:

$$\frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} = u(x, t) * (1 - u(x, t))$$

This simplifies to:

$$u(x, t + \Delta t) = u(x, t) + \Delta t * u(x, t) * (1 - u(x, t))$$

–$u = [u, u_{out}]$; this line adds a new column to the matrix with every new value of u_{out}.

–$plot(x, u_{out}, 'k-')$; this makes the graph. the 'k-' is for the color of the curve

–$pause(.01)$; this pauses the program in .01 second intervals. You can see the curve as the program is being executed

–End; this ends the proccess.

In this equation, we use an approximation of u_{xx}:

$$u_{xx} = \frac{u(x + \Delta h, t) + u(x - \Delta h, t) - 2u(x, t)}{\Delta h^2}$$

We use this approximation because of the Taylor expansions that follow:

$$f(x + \Delta x) = f(x) + f'(x)\Delta x + \frac{1}{2}f''(x)\Delta x^2$$

$$f(x - \Delta x) = f(x) - f'(x)\Delta x + \frac{1}{2}f''(x)\Delta x^2$$

If we add these two lines together, we obtain:

$$f(x + \Delta x) + f(x - \Delta x) = 2f(x) + f''(x) * \Delta x^2$$

There is a phase portrait associated with any autonomous ODE. A phase portrait arises from the combination of all the potential initial conditions. It is a geometric representation of the trajectories of the equation.

In the case of the logistics equation, we have two equilibrium points, u=0 and u=1. We know u=0 is unstable and u=1 is stable because of the graph of the equation $f(x) = x(1 - x)$

As you look at the graph, any point on the curve that crosses the x-axis going from positive y-values to negative y-values, as you move left to right, is a stable equilibrium point. Otherwise, it is unstable.

Knowing that u=0 is unstable and u=1 is stable helps us to see the solutions of the euqation. After drawing the phase portrait we can see that any initial conditions great than one will result in a solution approaching one. This also applies to any initial conditions in between zero and one.

As this is equation is most generally associated with population, negative initial conditions do not exsist. Therefore, every inintial condition greater than zero will eventually approach one.

I also studied what would happen if the initial conditions were altered. In the Matlab code for this equation, we have the initial conditions set to be $u_{out} = 2sin(x)'$. If we change these initial conditions, to $u_{out} = cos(x)'$ for example, the behavior of the solution changes a little bit, but in the end, all solutions do become asymptotic to the same number, in our case, one.

Applying this to biological invasion, this just means that no matter what the population size at the start, the population will always level off at the carrying capacity. Whether the population starts off with astronomically high numbers, or very minimal, it will most certainly approach the carrying capacity after a certain amount of time, t.

2.2 Heat equation

If we ignore the nonlinear term $f(x, u)$

$$\partial_t u = u_{xx}, \qquad x \in \mathbb{R},\ t > 0 \tag{2.9}$$

$$u = \begin{cases} 0, x < -1 \\ 1 + x, \leq x < 0 \\ 0 \leq x < 1 - x \\ x \geq 1. \end{cases} \tag{2.10}$$

$$u = 0,\ \text{when } x = \pm 10. \tag{2.11}$$

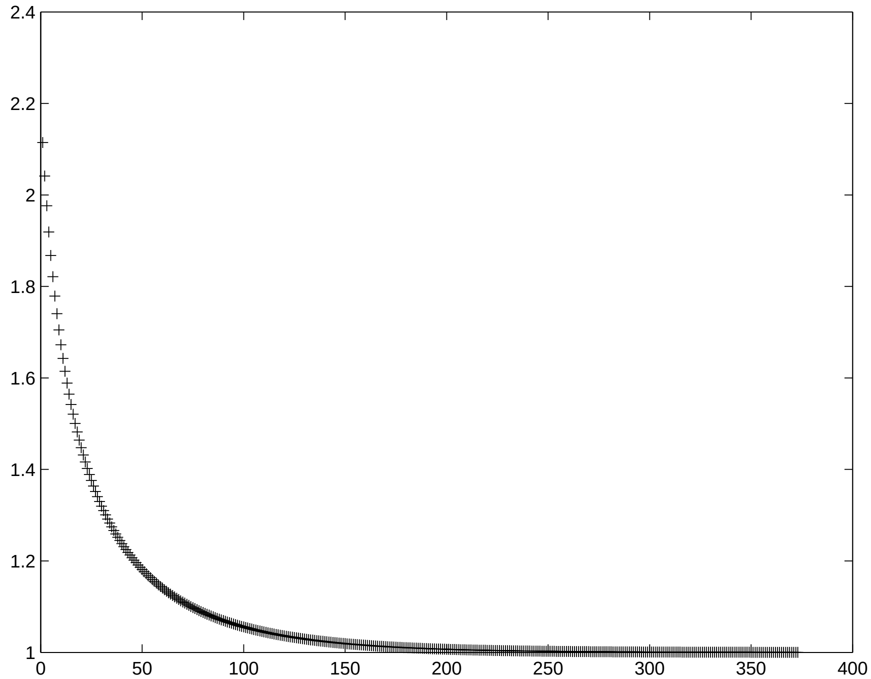


Figure 1: This is a trajectory from the logistics equation with the altered initial conditions. As you can see, the begining of the curve is much different, but it still tends toward one for large time, t.

The heat equation is a partial differential equation that describes the distribution of heat in a given area in a given time interval.

Equilibrium solutions for this equation are solutions for which there is no heat flow.

Since this a PDE we can also set boundary conditions. Say we fix a rod with one end on a block of ice and the other end attached to a heater. If the minimun temperature of the ice is u(-10,t)=0 and the maximum of the heater is u(10,t)=1, then these are our boundary conditions.

As t approaches infinity, the heat stops moving. At this point, every point on the rod is a certain constant temperature. The temperature of the graph versus the position can be represented linearly. The middle of the rod will be exactly the middle temperature of the whole rod.

I find it to be particularly interesting that in a system with a rod connected to a heater on one end and some sort of refrigerator on the other, the interior points on the rod will not not excede the temperature of the heater and will not drop below the temperature of the refrigerator. This is known as the maximum heat principle.

The maximum principle, as it pertains to mathematics, states that if f is a harmonic function, then it will not have a local maximum within the domain of f. This function will either be constant, or for any point x inside the domain of f, there will exsist other points arbitrarily close to x such that there is no true local maximum.

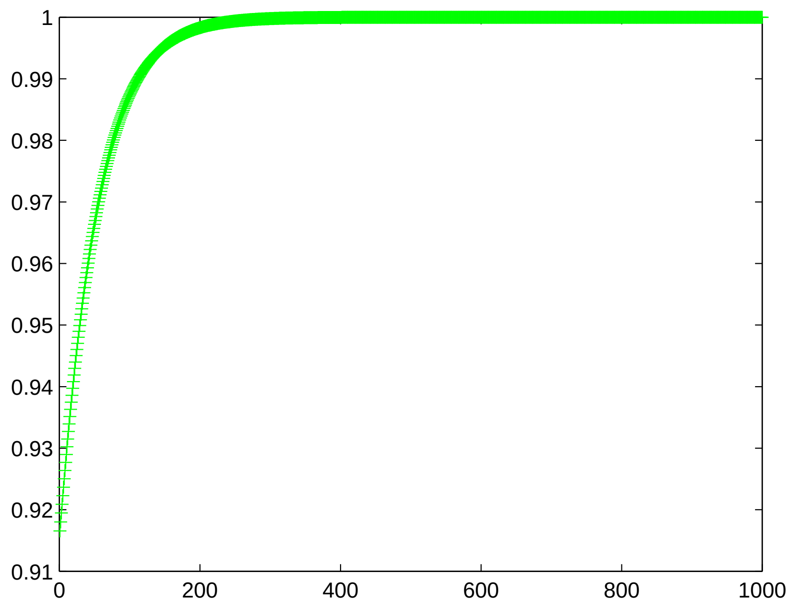


Figure 2: This is a graph of a trajectory of the solutions to the logistics equation. As you can see, any initial conditions inbetween zero and one will all eventually become asymptotic to one.

Also, any two solutions of this equation can be combined to form another solution. This is known as the principle of superposition. Any linear combination of two solutions is also a solution. Also, provided that both of the original solutions satisfy the given boundary conditions, then the linear combination of those two solutions will also satisfy the boundary conditions.

Writing this Heat Equation into Matlab we obtain the following lines of code:
– $\varepsilon = 2$; This assigns the value of 2 to the variable epsilon
– $\alpha = \varepsilon$; This makes the value of alpha equal to the value of epsilon

– $\beta = 1$; This assigns the value of 1 to the variable beta
– $dh = 0.5$;
– $size_x = 10$; This tells us that x ranges from -10 to 10
– $nx = round(\frac{2size_x}{dh} + 1)$; This is the number of points
– $x = linspace(-size_x, size_x, nx)$; This is the location of the points. x ranges from -10 to 10 and nx is the increment
– $\chi_{solid} = 0 * x'$; This means that for any x, χ_{solid} will be zero. x' means it is the transpose of the x vector
– $dt = .1$; This is the time step
– T=100,000; This is the terminal time.
– $N = round(\frac{T}{dt})$; This is the number of steps. It is the difference between the terminal time and the time step, rounded to the nearest whole number

– $u_{out} = 2 * max(0, sin(x))'$ $u_{out}(nx) = 1$ $u_{out}(1) = 0$ These are the initial conditions

– $u = []$; This line makes the matrix. It makes room for the first value.

– $E = gallery('tridiag', nx, -1, 1, 0)$; This makes a matrix with nx-many rows and nx columns. There are ones on the main diagonal and negative one's on the diagonal directly below it. Everywhere else is zero.

– $D = \varepsilon$; This makes the value of D equal to the value of epsilon.

– A=-E*D*E'; -E is the same layout as E, except all of the ones are negative and all of the negative ones are positive. D is just the number 2, in our case and E' is simple the transpose of E. It has ones on the main diagonal with negative ones on the diagonal above it. A is just the product of these matrices.

– A(1,1)=0

– A(1,2)=0; These lines force the first row of the matrix to be entirely zero.

– A(nx,nx)=0

– A(nx,nx-1)=0; These two lines force the last row of the matrix to be entirely zero.

– $id = eye(nx)$; This is the identity matrix of size nx, which is 41 in our case.

– $S = inv((id - A) * \frac{dt}{dh^2})$; This line is just the inverse of the identity matrix minus the matrix A that we constructed, multiplied by dt over dh squared.

– $u_{out} = S * u_{out}$; ?

– for $i = 1 : 1 : N$; i is the counter in this process and this makes the process run until it reaches N. It goes in increments of one.

– $u_{out} = S * u_{out}$

– $u = [u, u_{out}]$; This line adds a new column to the matrix with every new value of u_{out}

– $plot(x, u_{out}, ' k-')$; This plots the graph of the equation.

– $pause$; this allows you to pause the function as it is changing.

– end; This ends the process.

The same proccess of altering initial conditions can be applied to the heat equation. As previously stated, heat tends to move from warmer areas to cooler areas, in an attempt to create equilibrium. So, no matter how many "warmer" areas there are or how many "cooler" areas, the system will eventually reach an equilibrium.

As it applies to biological invasion, this simply means that no matter what the size of the area that a certain species lives in, there will be diffusion. In the example of our mice living on an island, it would be like saying no matter what the size of the island, the mice will eventually diffuse over the entirety of the island, provided they are physically capable. It can also be interpreted as no matter how large the larger groups are, and how smaller the smaller ones, they will always diffuse together and the larger will become smaller and the smaller will become larger.

2.3 Reaction-Diffusion equation

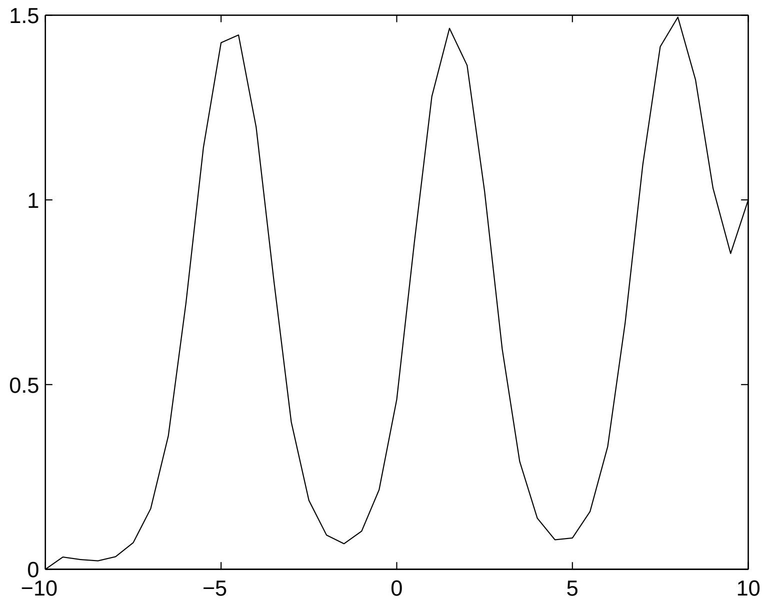


Figure 3: This is the beginning, when time t=0. As you can see by the maxima and minima, the temperature of the rod is not constant.

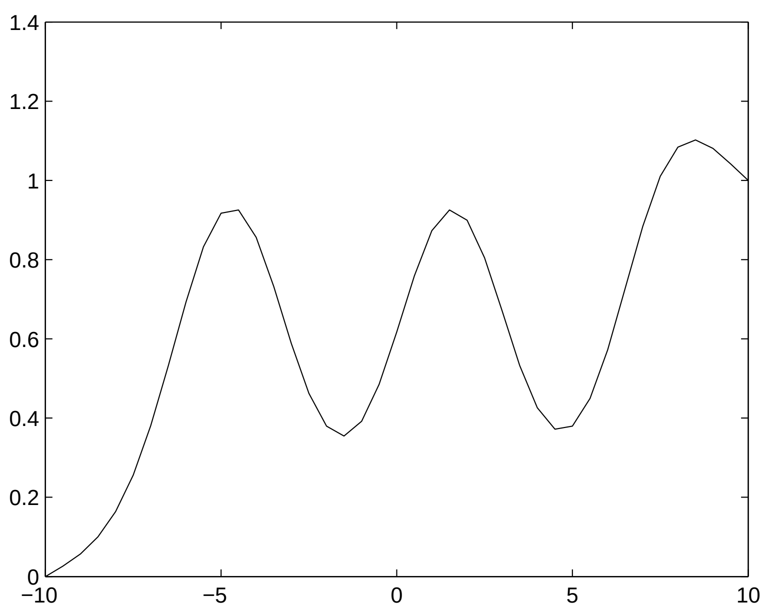


Figure 4: As t begins to approach infinity, the maxima begin to drop and the minima begin to climb.

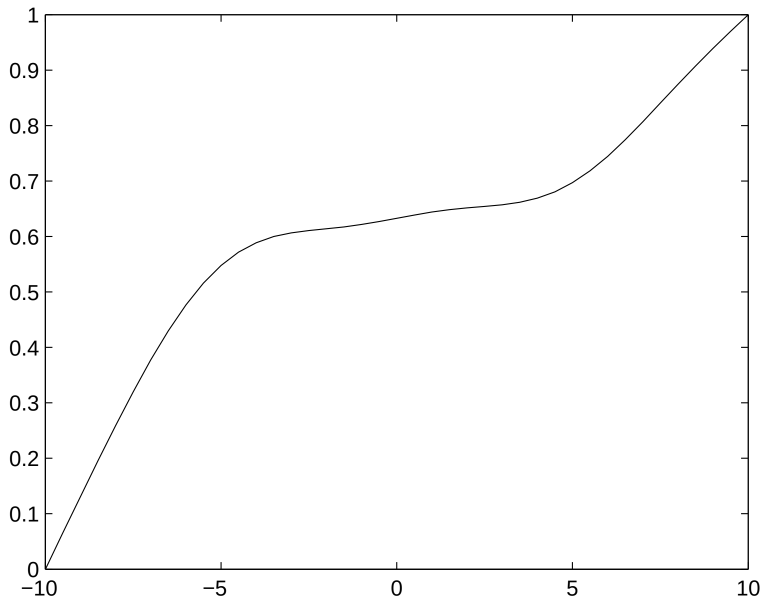


Figure 5: After a certain amount of time has passed, the number of maxima and minima greatly decreases.

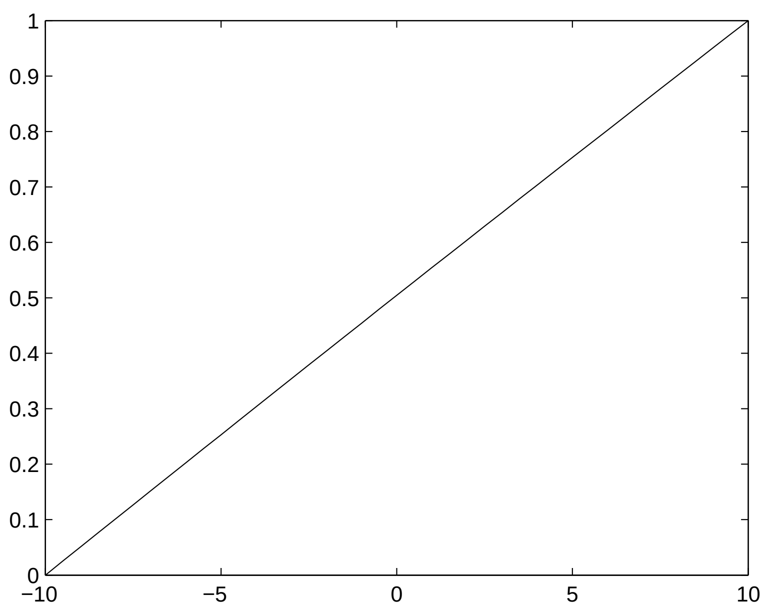


Figure 6: Finally, as t is nearing infinity, the solution is linear.

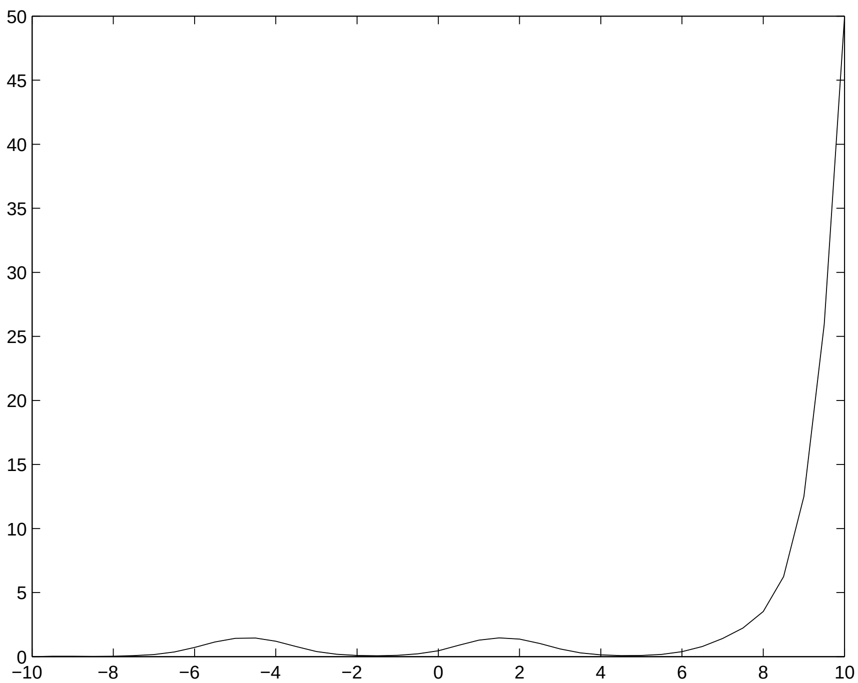


Figure 7: Here, I altered the initial conditions of the Heat equation. Instead of having the end points be zero and one, I chose them to be zero and fifty. The curve looks different from the originally one, however you can see that it does indeed have the same general form.

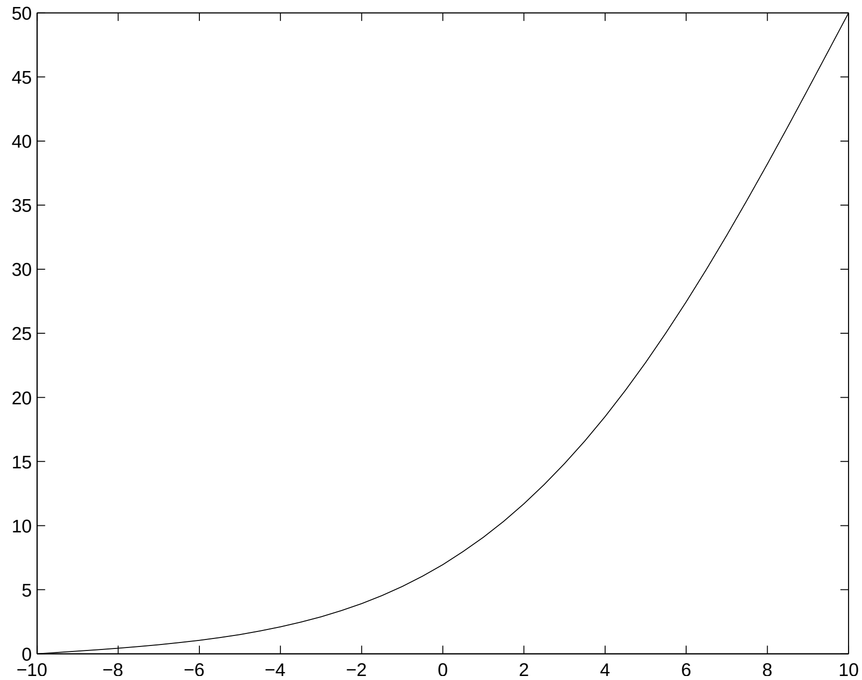


Figure 8: As you can see, as time continues, the "bumps" begin to even out as the warmer areas become cooler and the cooler areas become warmer. Analogous to the other graph, the curve is begining to become linear.

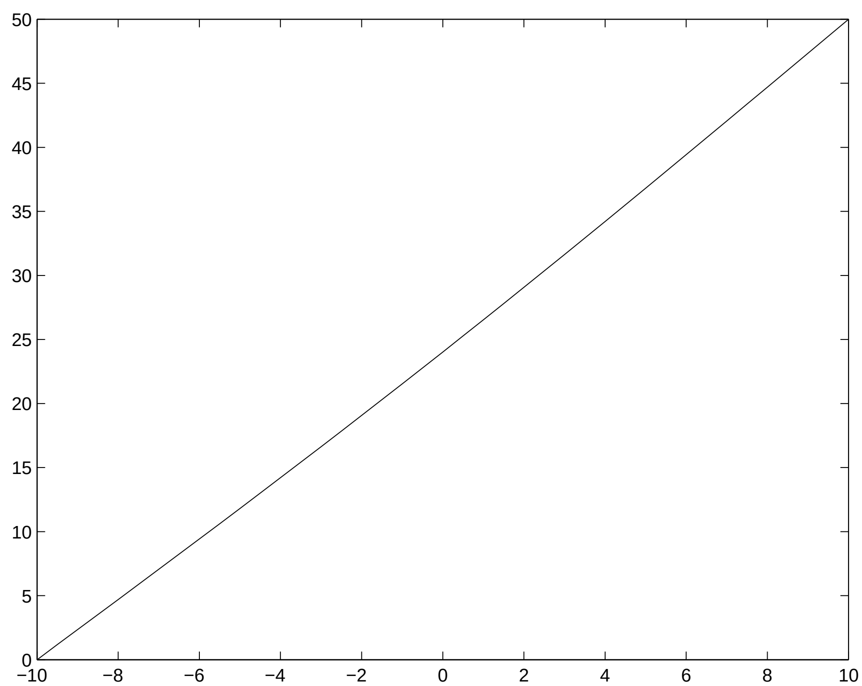


Figure 9: This is the graph for the altered heat equation after a very long time, t. As you can see, the curve has become linear and the system has reached equilibrium.

Here we combine the heat equation and the logistics equation together and study the reaction-diffusion equation:

$$\partial_t u = u_{xx} + u(1-u), \qquad x \in [-10, 10]\ t > 0 \tag{2.12}$$

$$u = \begin{cases} 0, x < -1 \\ 1 + x, \leq x < 0 \\ 0 \leq x < 1 - x \\ x \geq 1. \end{cases} \tag{2.13}$$

$$u = 0, \text{ when } x = \pm 10. \tag{2.14}$$

The reaction-diffusion equation is an equation that combines both the heat equation and the logistics equation. One application of this equation is the study of population density in animals.

Given a community of animals, the logistics part of this equation represents the population growth of this community. When the population of a species is very small, there is very littly competition. When this happens, the population will grow without bound because there is nothing keeping it from growing. However, as the population grows, competition does begin to exsist.

When comptetition takes effect, the population is no longer growing without bound. The number of animals in this community will eventually level off. This happens because of the carrying capacity. In terms of population, only a certain number of animals can comfortably live in a certain area.

The diffusion part of this equation represents the fact that animals will disperse over their land as their population grows. It can describe how the population is expanding, without population growth.

Putting the two parts together, we obtain the reaction-diffusion equation. As previously stated, this equation can be used to describe the population growth and range of a certain species. The typical solution to this equation is described as the traveling wave solution, described in the next section.

2.4 Traveling wave solutions of the Reaction-Diffusion equation on real line

As described before, the reaction-diffusion equation is the combination of the diffusion equation and the logistic equation. I found particular solutions of this equation by setting $u(x,t) = V(T)$, $T = x - ct$ and rewriting it as a first order differential equation.

$$-cV'(T) =''(T) + V(T)(1 - V(T))$$

With $x_1 = V$ and $x_2 = V'$, we obtain the following:

$$x_1' = V$$

$$x_2' = -cx_2 - x_1(1 - x_1)$$

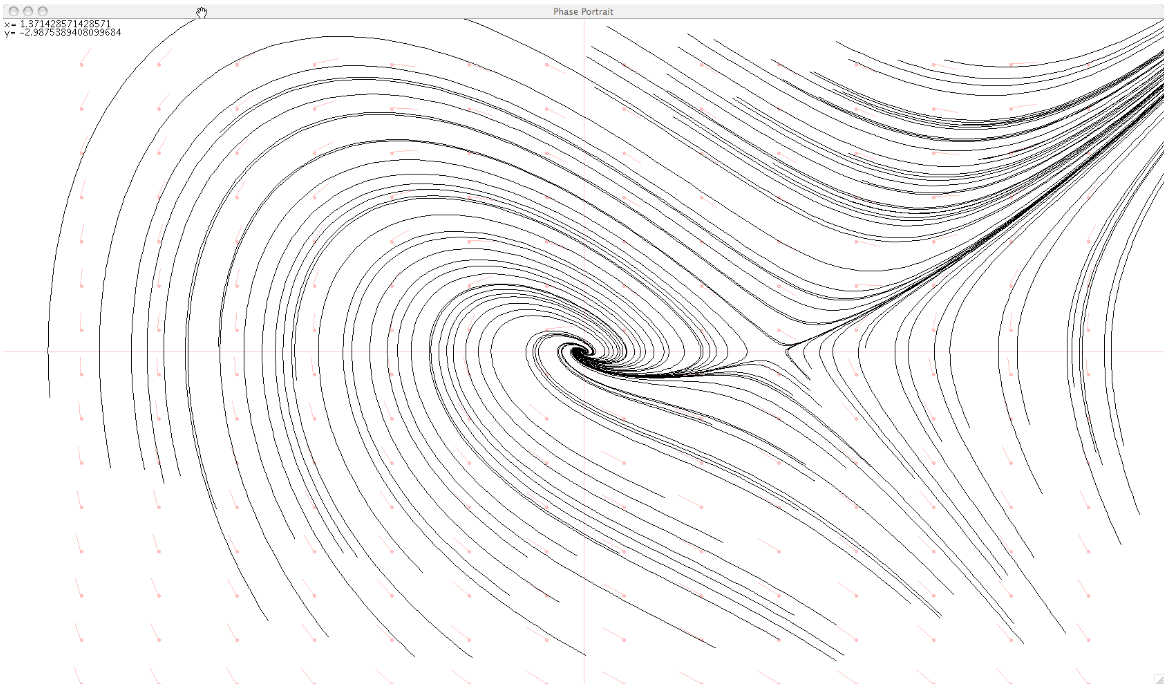


Figure 10: This is the phase portrait of the solutions with $c < 2$. As you can see, there is a stable spiral and a saddle point.

First, we need to find the equilibrium points. To do this we set both equations equal to zero.

$$0 = x_2$$

$$0 = -cx_2 - x_1 + x_2^2$$

This gives us two critical points, (1,0) and (0,0). Upon further analysis of these equilibria, we discover that, if $c < 2$, we have a saddle and a stable spiral.

However, if $c > 2$, there exists a heteroclinic orbit, $\gamma(T)$, such that $\gamma(T) - > (0,0)$ as T approaches infinity. Also, $\gamma(T) - > (1,0)$ as T approaches negative infinity. A heteroclinic orbit, simply put, is a trajectory on a phase portrait that joins two different equilibrium points. As you start from the (0,0) equilibrium point, the trajectory goes to (1,0). The same thing happens from the other point. As you start fom (1,0), the trajectory goes to the point (0,0). Another graph with a heteroclinic orbit is the graph of the phase portrait for the pendulum.

As you can tell by these graphs, the wave front travels in the positive x direction at some speed c.

The following graph of the trajectory describes the heteroclinic orbits discussed earlier.

These travelling wave solutions move at a constant speed, c, and they do not change their shape or form as they move along the x-axis.

In continuing my study of the reaction-diffusion equation, I also studied a different form of the equation. Instead of the previous

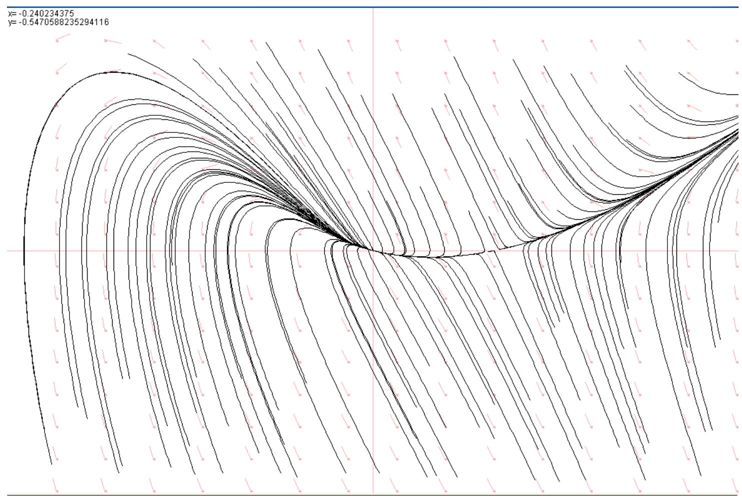
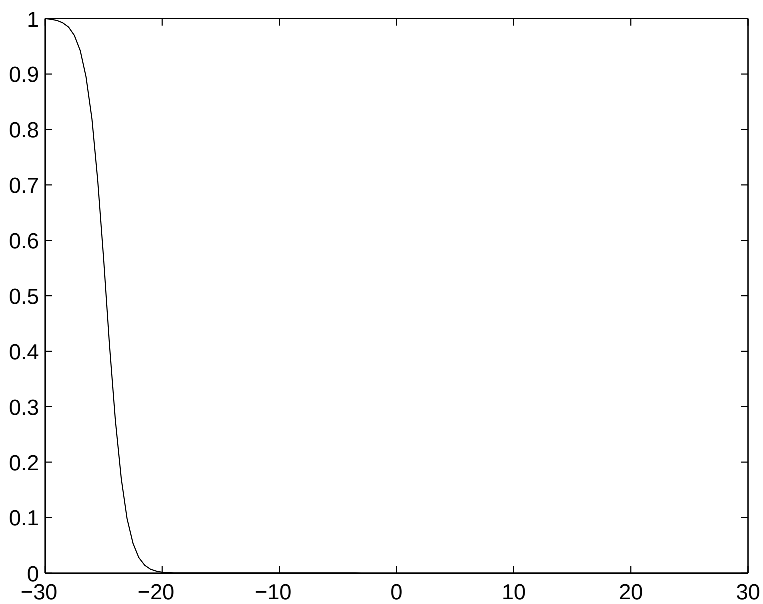


Figure 11: Here, we have $c > 2$ and thus there is a heteroclinic orbit. As you can see, trajectories approach (0,0) as T approaches infinity and trajectories approach (1,0) as T approaches negative infinity.



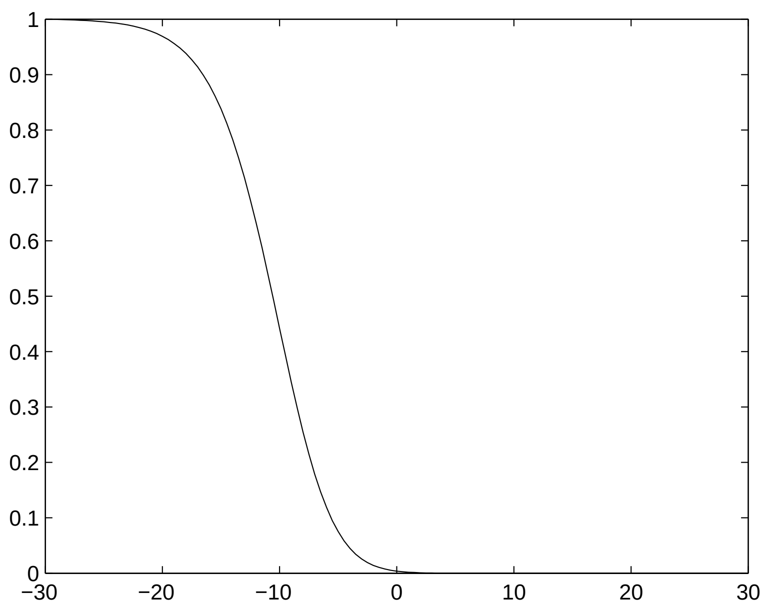


Figure 12:

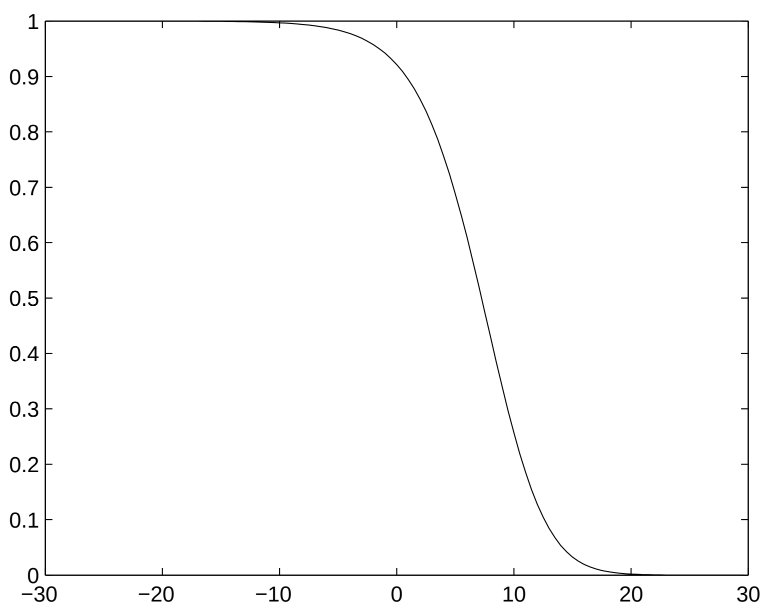


Figure 13:

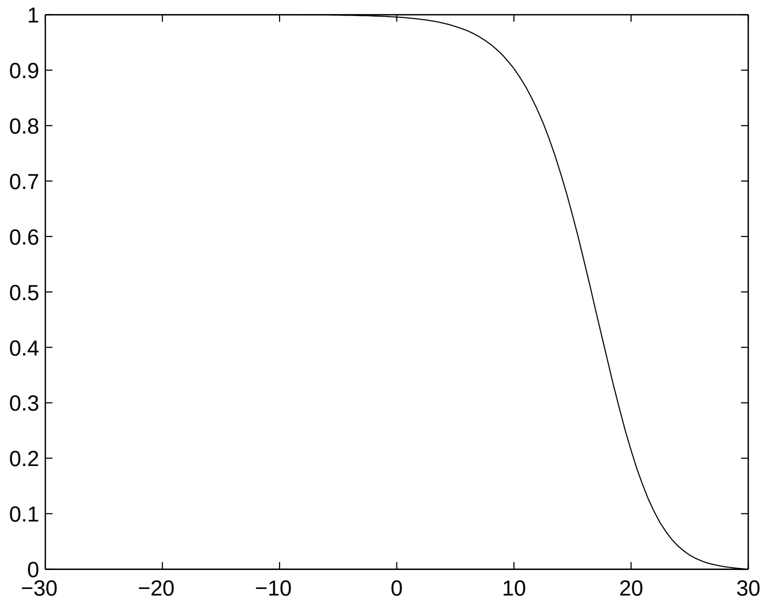


Figure 14:

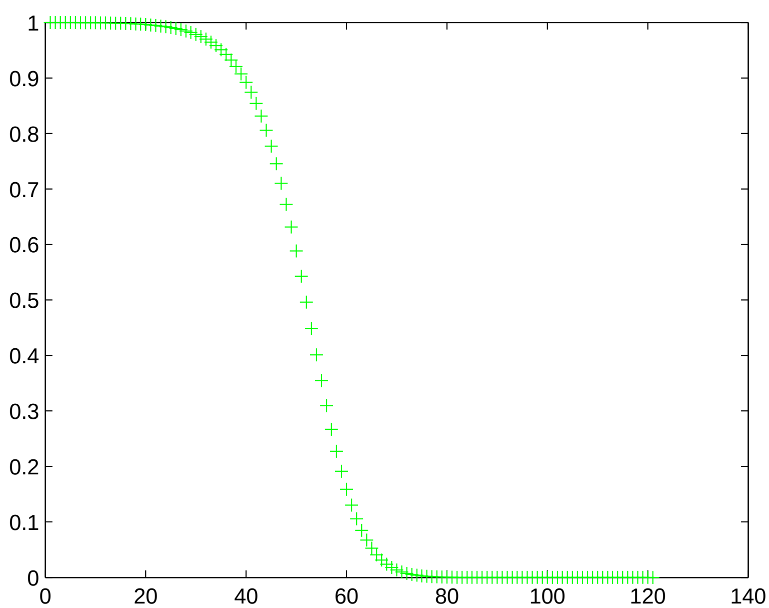


Figure 15: This graph shows that as T approaches infinity, this trajectory is approaching zero.

$$u_t = u_{xx} + u(1-u)$$

I decided to study what would happen if we squared the u in the logistics portion of the equation as such:

$$u_t = u_{xx} + u(1-u^2)$$

First, I rewrote this equation just as it was done earlier. Previously, we had set $u(x,t) = V(x-ct) = V(T)$. Using this equality and subbing into this new equation, we arrive at the equation

$$-cV'(T) = V''(T) + V(T)(1 - V(T)^2)$$

In order to solve this manually, I set $V = x_1$ and $V'(T) = x_2$. Subbing this in we get a system of two first order differential equations.

$$x_1' = x_2$$
$$x_2' = -cx_2 - x_1(1 - x_1^2)$$

From here, I solved this system just as I did the other one, I started by setting both equations equal to zero in order to find the equilibrium points. Upon doing this, you arrive at the following critical points: (-1,0), (0,0), (1,0). From here I found the eigenvalues by plugging these points into the matrix of partial derivatives. For the points (-1,0) and (1,0), you arrive at the same characteristic equation for both equilibrium points. This gives you the eigenvalues of

$$\lambda_1 = \frac{-c + (c^2+8)^{\frac{1}{2}}}{2} > 0$$

and

$$\lambda_2 = \frac{-c - (c^2+8)^{\frac{1}{2}}}{2} < 0$$

Since λ_1 is always > 0 and λ_2 is always < 0, this means that these two points are saddle points no matter what the given value of c. The last critical point, however, is much different. The characteristics equation yields the following two eigenvalues:

$$\lambda_1' = \frac{-c + (c^2-4)^{\frac{1}{2}}}{2}$$

and

$$\lambda_2' = \frac{-c - (c^2-4)^{\frac{1}{2}}}{2}$$

These two eigenvalues are not strictly positive or strictly negative. For all $c > 2$, $\lambda_1', \lambda_2' < 0$. For $c < 2$, (0,0) is an unstable node. This creates two different phase portaits.

However, for all $c < 2$, $\lambda_1', \lambda_2' > 0$. Thus, for $c > 2$, this point, (0,0), is a stable node; it is a sink.

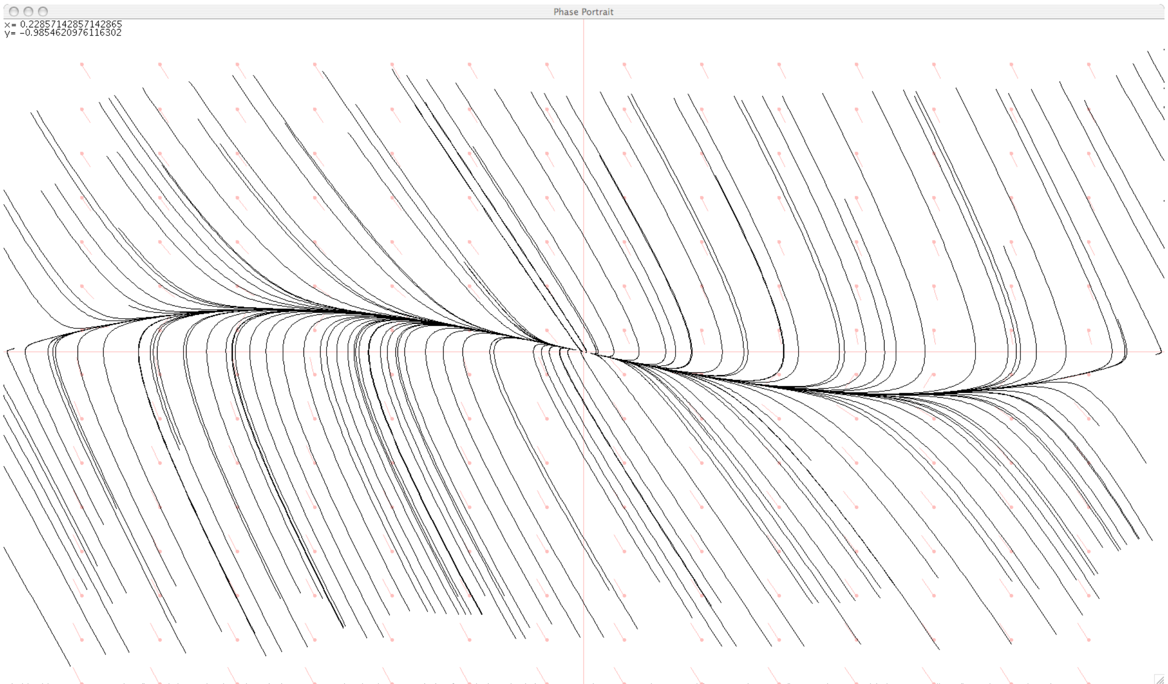


Figure 16: This is the phase portrait for all $c > 2$. The graph is zoomed in to the origin, as that is our area of interest. As you can see, this point is unstable as all solutions move away from zero.

In terms of biological invasion, there does not appear to be much difference in the prediction of population growth. Even though there is the addition of the third critical point, (-1,0), it does not mean much in terms of population. This is true because population numbers are strictly positive. It is impossible for a species to be negative in terms of population. The following graphs prove this statement.

2.5 Reaction-Diffusion equation in heterogeneous media

We now study the Reaction-Diffusion equation in heterogeneous media.

In terms of biological invasion, heterogeneity in habitat could affect the spread of a species. Sticking with the mice example, we can see that if the mice are dispersing through the forest, they could be facing many different kinds of terrain. For mice, land forms such as rivers and mountain and even wide-open valleys could create migration problems. These types of situations could potentially cause the mice to move very slowly. In contrast, if the mice were moving through the forest, protected from the eyes of a predator, the mice could possibly be moving at a much faster rate.

With this kind of information, we can predict whether an animal will be able to invade a certain terrain. We can also predict the rate at which these animals can invade. The definition of rough terrain versus easy terrain varies by species, but no matter what the animal or plant, rough terrian will always indicate slower movement and easy terrain will be indicative of quicker movement.

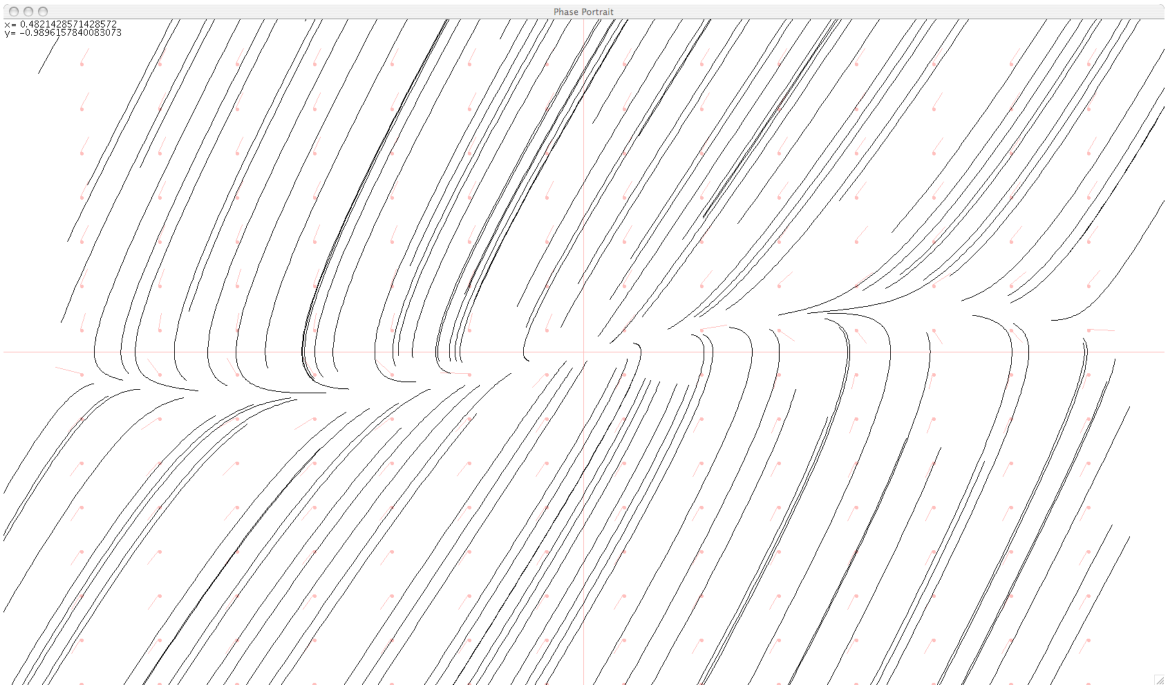


Figure 17: This is the phase portrait for all $c < 2$. Again, the graph is focused on the origin. This point, unlike the previous one, is stable. It is a sink because all solutions approach this equilibrium point.

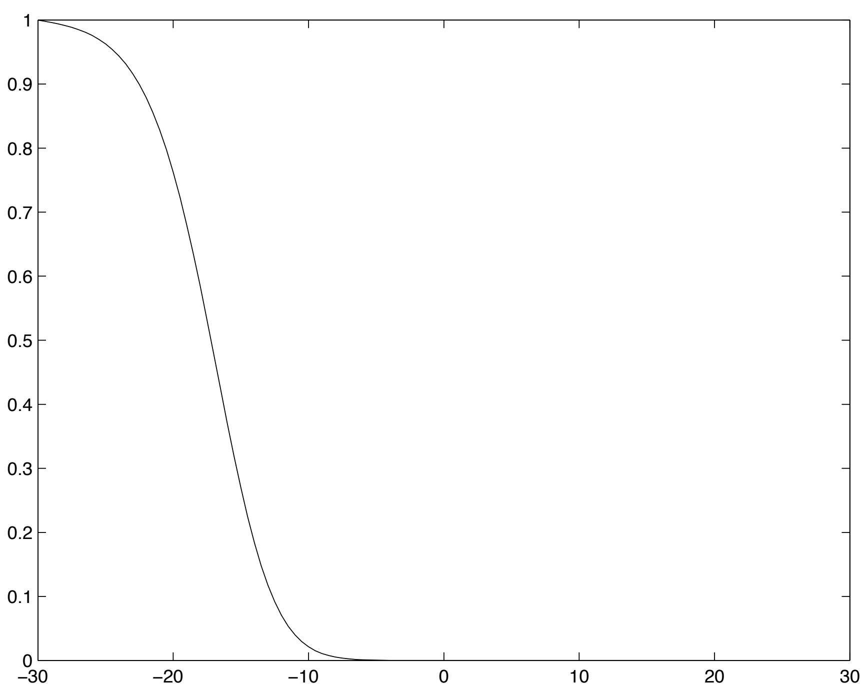


Figure 18: This is the graph of the reaction-diffusion equation with the logistics portion as such: $u(1 - u)$.

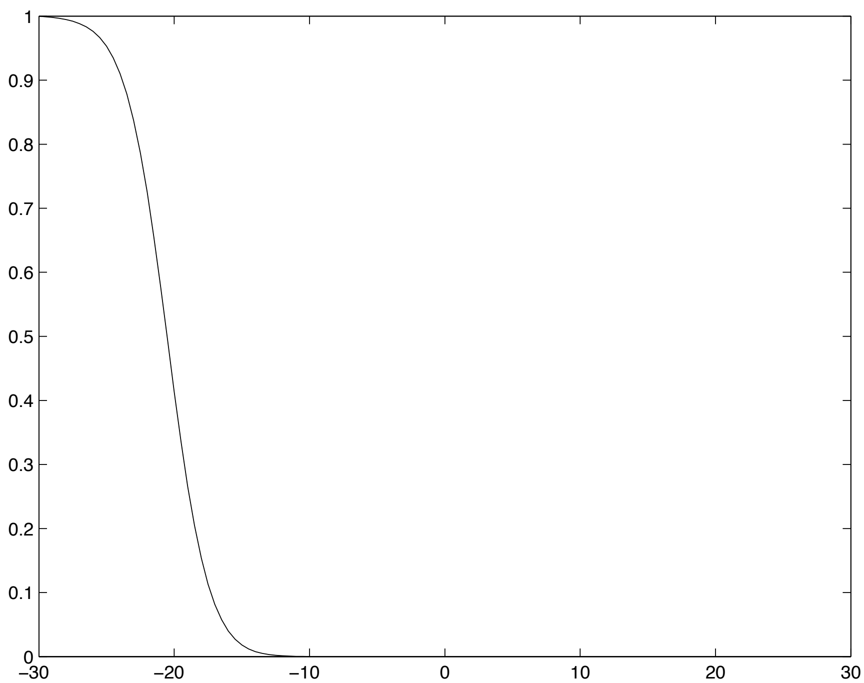


Figure 19: This is the graph of the reaction-diffusion equation with the logistics portion altered to be $u(1-u^2)$. As you can see, the graph is quite similar to the first one. The main visible difference seems to be the slope of the wave, which would be expected.

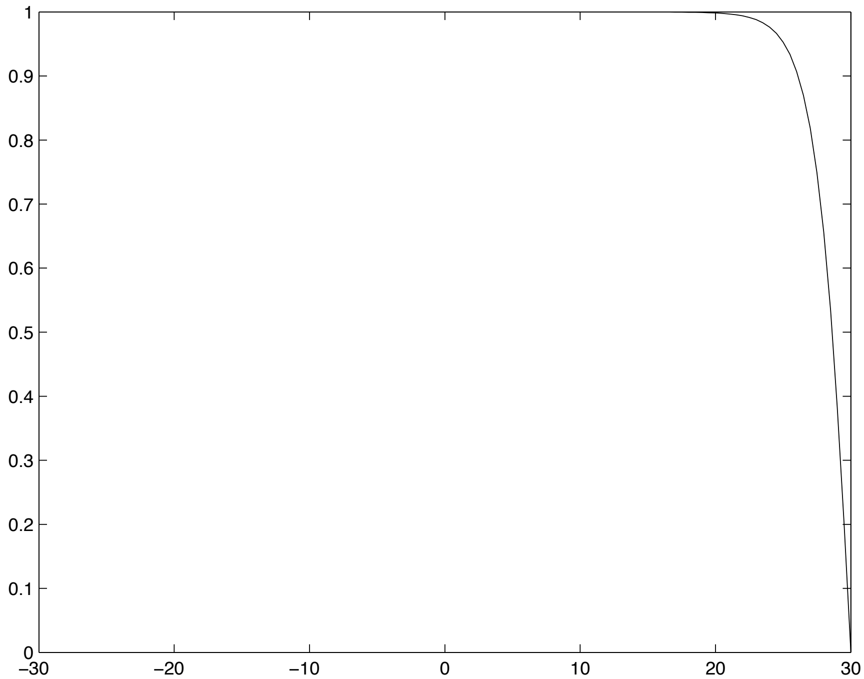


Figure 20: This graph shows the wave after a very long time t, with the original reaction-diffusion equation.

Changing the initial conditions in this equation, you can see that the effect is similar to changing the initial conditions of the other equations. Here, I changed the initial population number to three, instead of one. The outcome of this change was the same as before. Despite the changes, the population still hovered around the carrying capacity, one, for large time, t.

In terms of biological invasion, the original graph represented the different types of terrain a species would encounter in every day life. By changing these initial conditions, we are basically just changing the population size and the type of terrian they encounter. As you can see from the graphs below, this situation is very similar to the original one.

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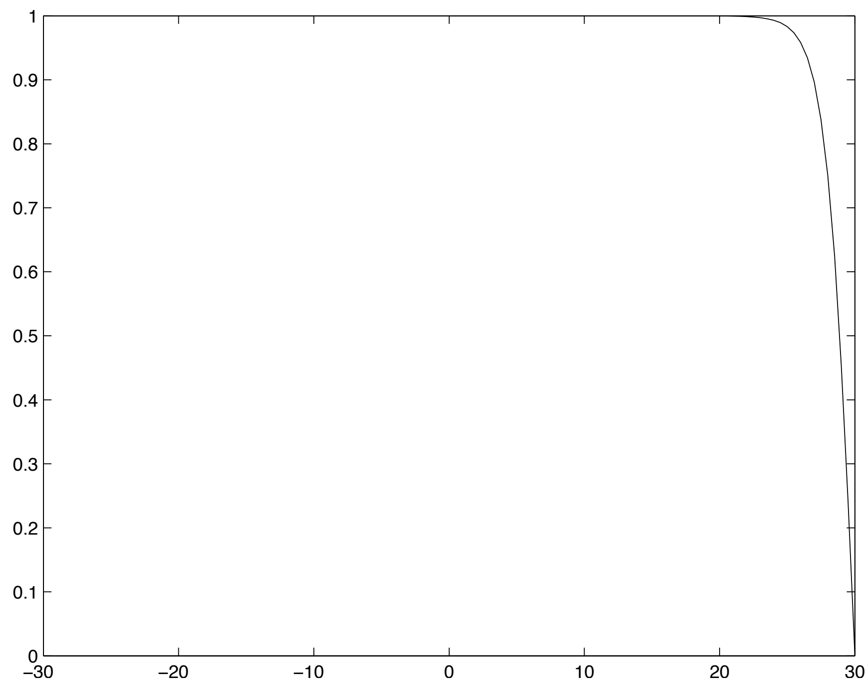

Figure 21: This graph, for the adjusted euqation, shows a very similar situation for some large t. Both graphs simply illustrate that after a long time, when a species has dispersed and grown, eventually the population will be stable in location and hover around the carrying capacity, which is one in our case.

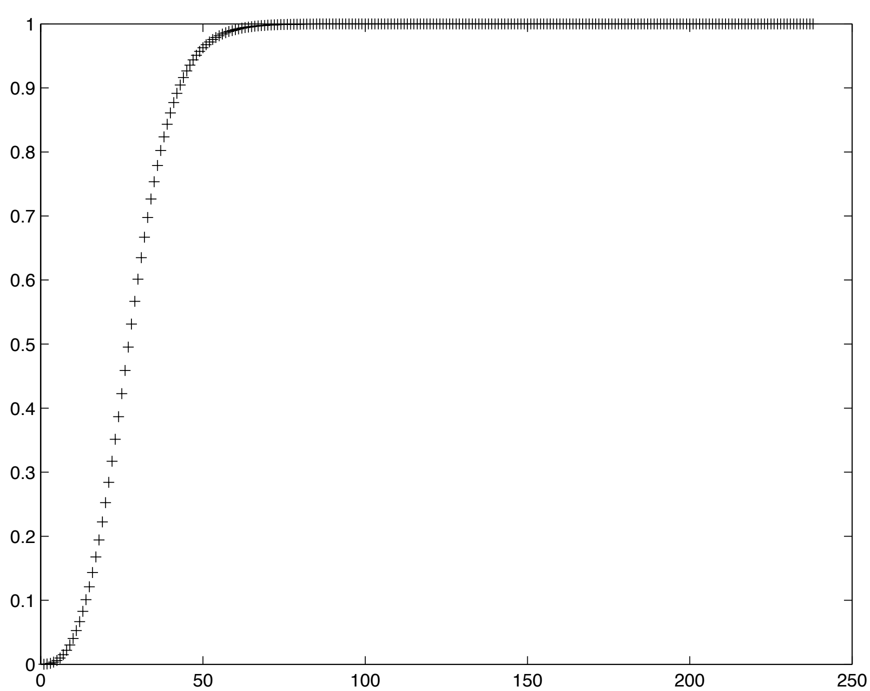

Figure 22: Here we have a graph of a trajectory from the first equation.

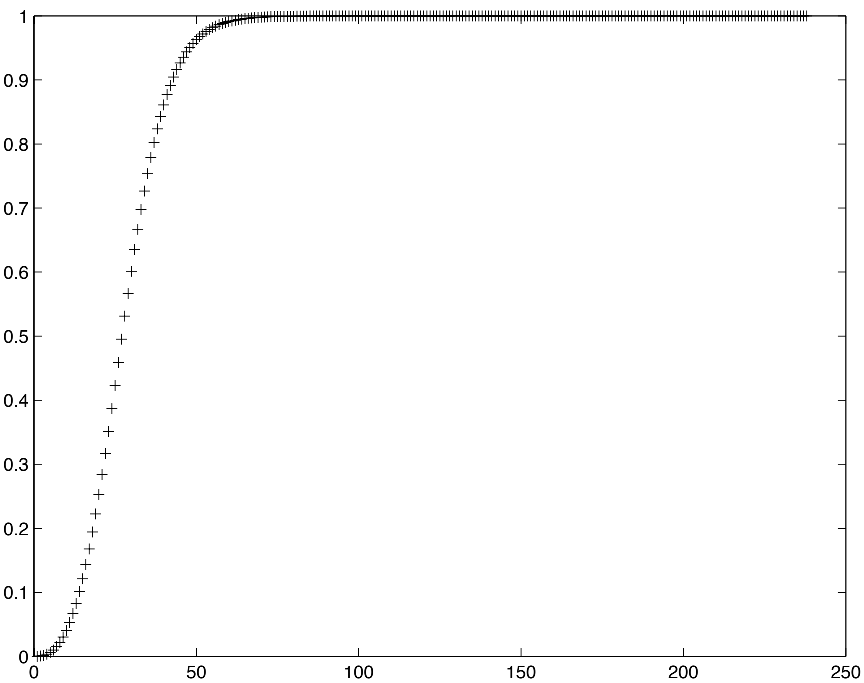

Figure 23: This trajectory is from the second, altered equation. As you can see, both trajectories approach and be asymptotic to one. No matter what the initial conditions or the diffusion, the population will always hover around the carrying capacity for very large time t.

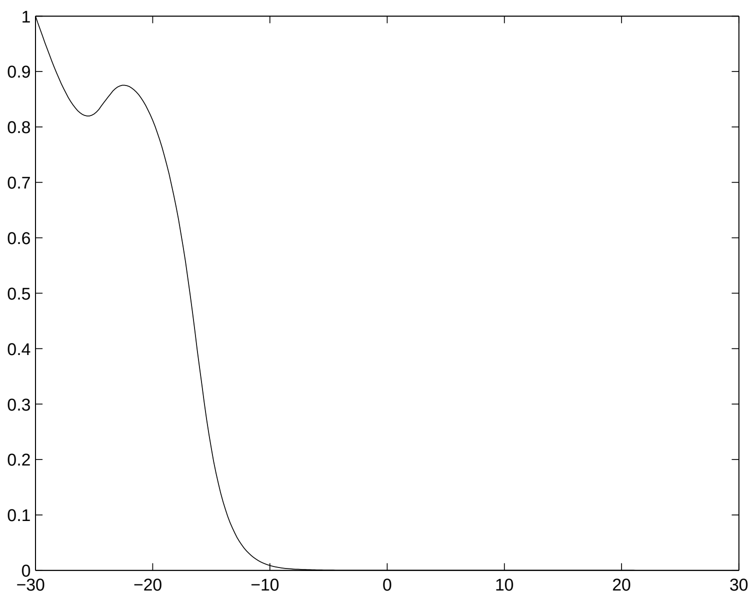

Figure 24: This graph, in our application, shows areas of rough terrain and easy terrain. The peaks and valleys can represent areas that are difficult or easy to travel through.

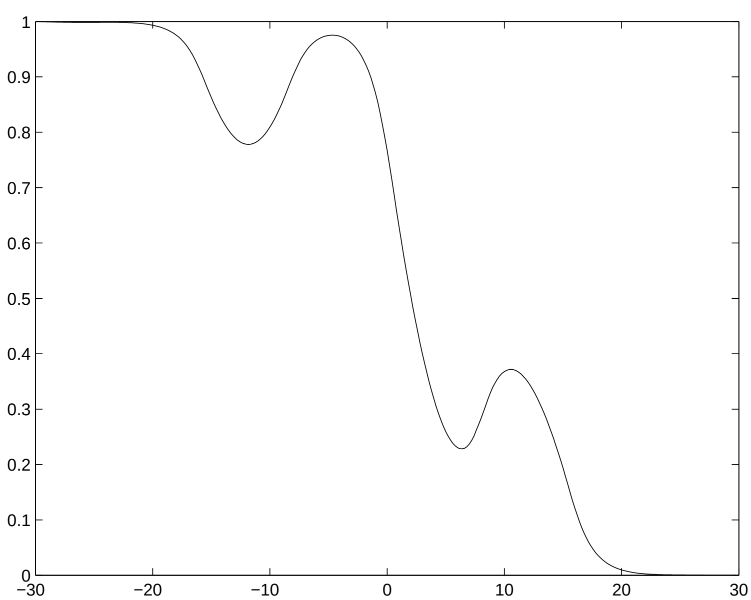

Figure 25: As you can see, as time increases, the maxima and minima begin to change, or attempt to reach an equilibrium.

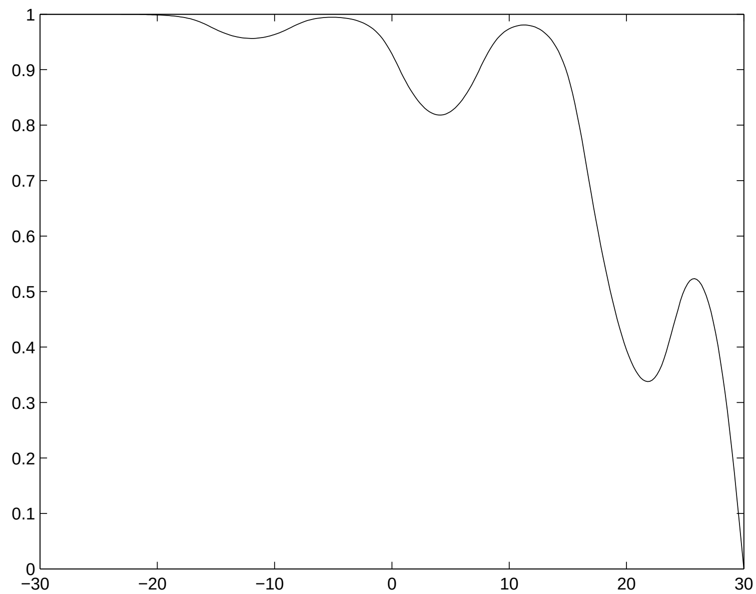

Figure 26: Here, the maxima and minima are still changing.

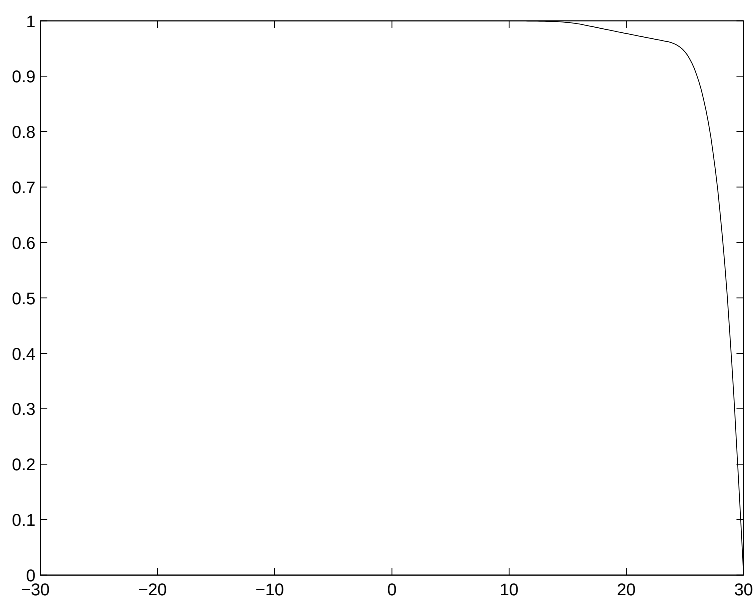

Figure 27: Finally, after a very long time, t, some sort of an equilibrium is reached.

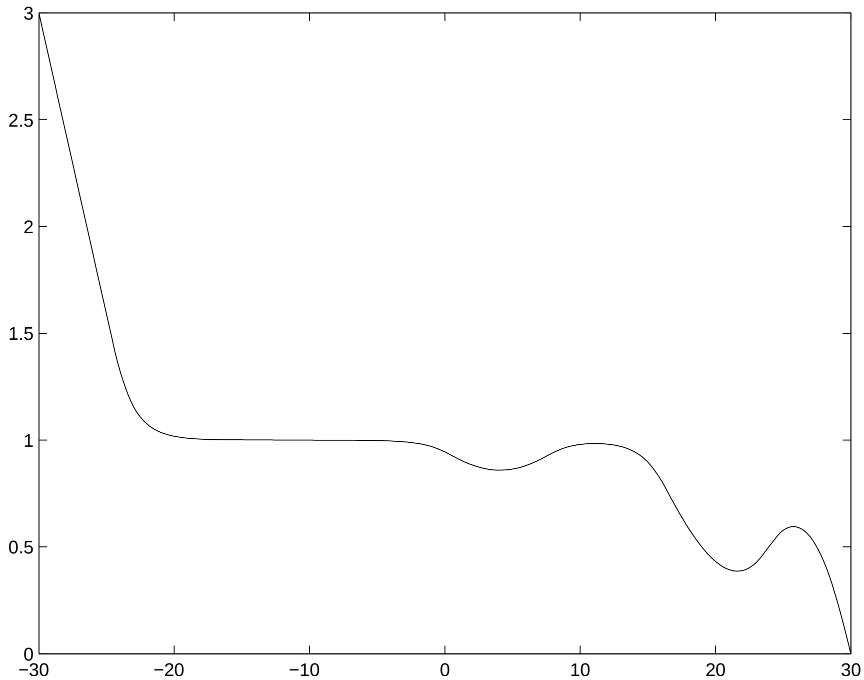

Figure 28: Here is the graph of the reaction-diffusion equation in heterogeneous media with the altered initial conditions. It is obvious that the curve is very similar to the original one with the exception of the beginning part. This is the part effected by the change in initial conditions.

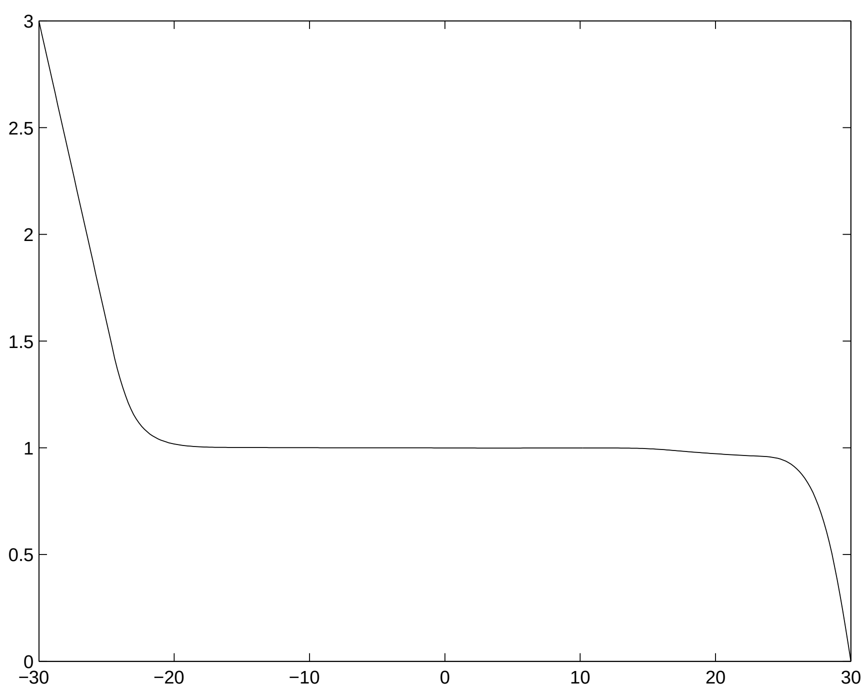

Figure 29: Just as before, the system reaches an equilibrium after a long time, t.

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